Pseudorandom Generators

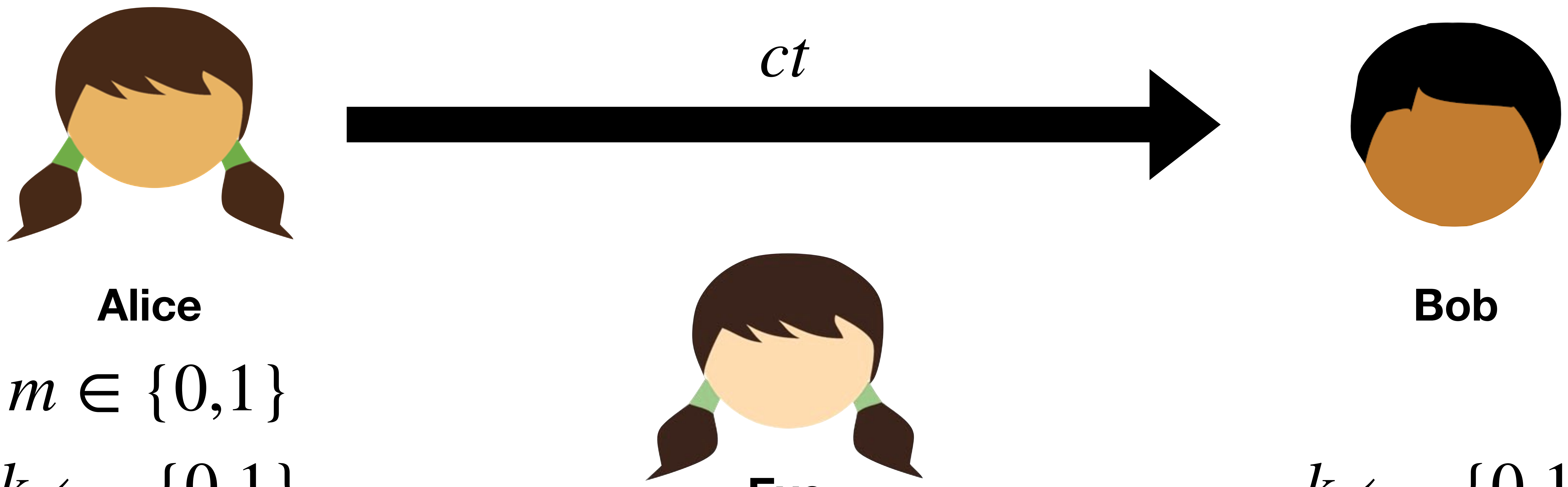
CS/ECE 407

Today's objectives

Describe pseudorandomness/pseudorandom generators

Define negligible functions

Understand security of PRGs



ct

Alice

$m \in \{0,1\}$

$k \leftarrow_{\$} \{0,1\}$

$ct \leftarrow m \oplus k$

Eve

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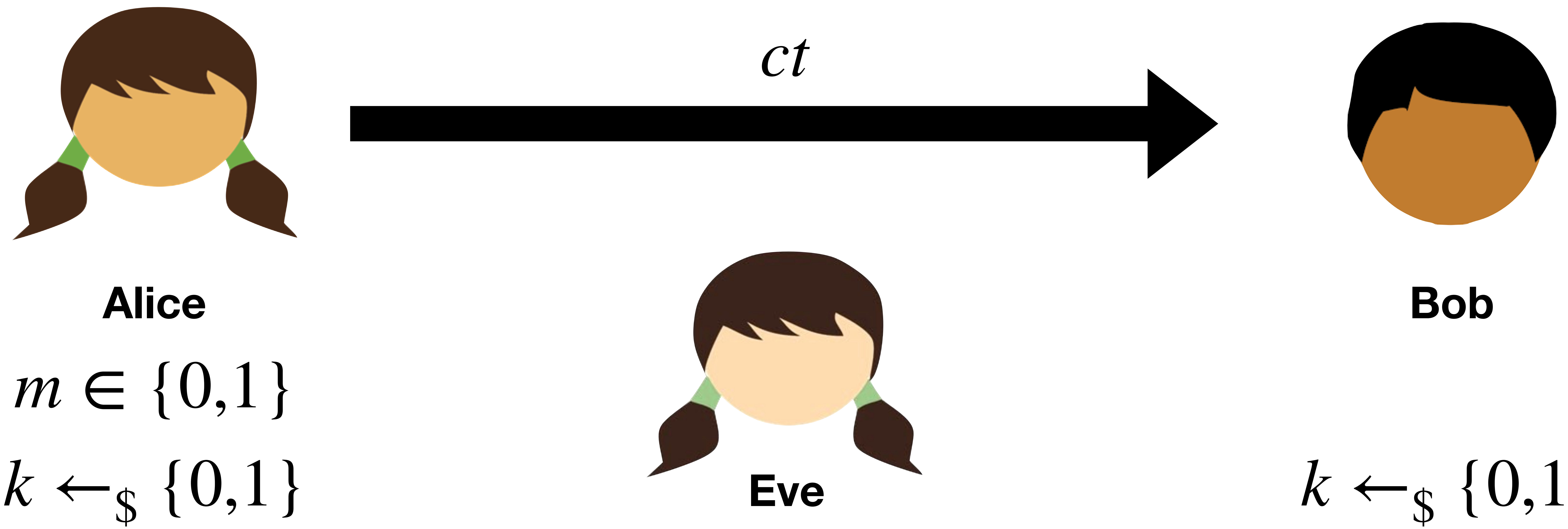

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Question: *what if Alice wants to send more than one bit?*

Perfect Secrecy:

For every pair of messages $m_0, m_1 \in \mathrm{M}$ and every cipher text $c \in \mathrm{C}$:

$$\Pr_{k \leftarrow \mathrm{K}}[\ Enc(k, m_0) = c\ ] = \Pr_{k \leftarrow \mathrm{K}}[\ Enc(k, m_1) = c\ ]$$

Theorem [Shannon 1949]: *Any cipher achieving perfect secrecy requires that* $|\mathrm{K}| \geq |\mathrm{M}|$*.*

“If we want to encrypt more stuff, we need more randomness”

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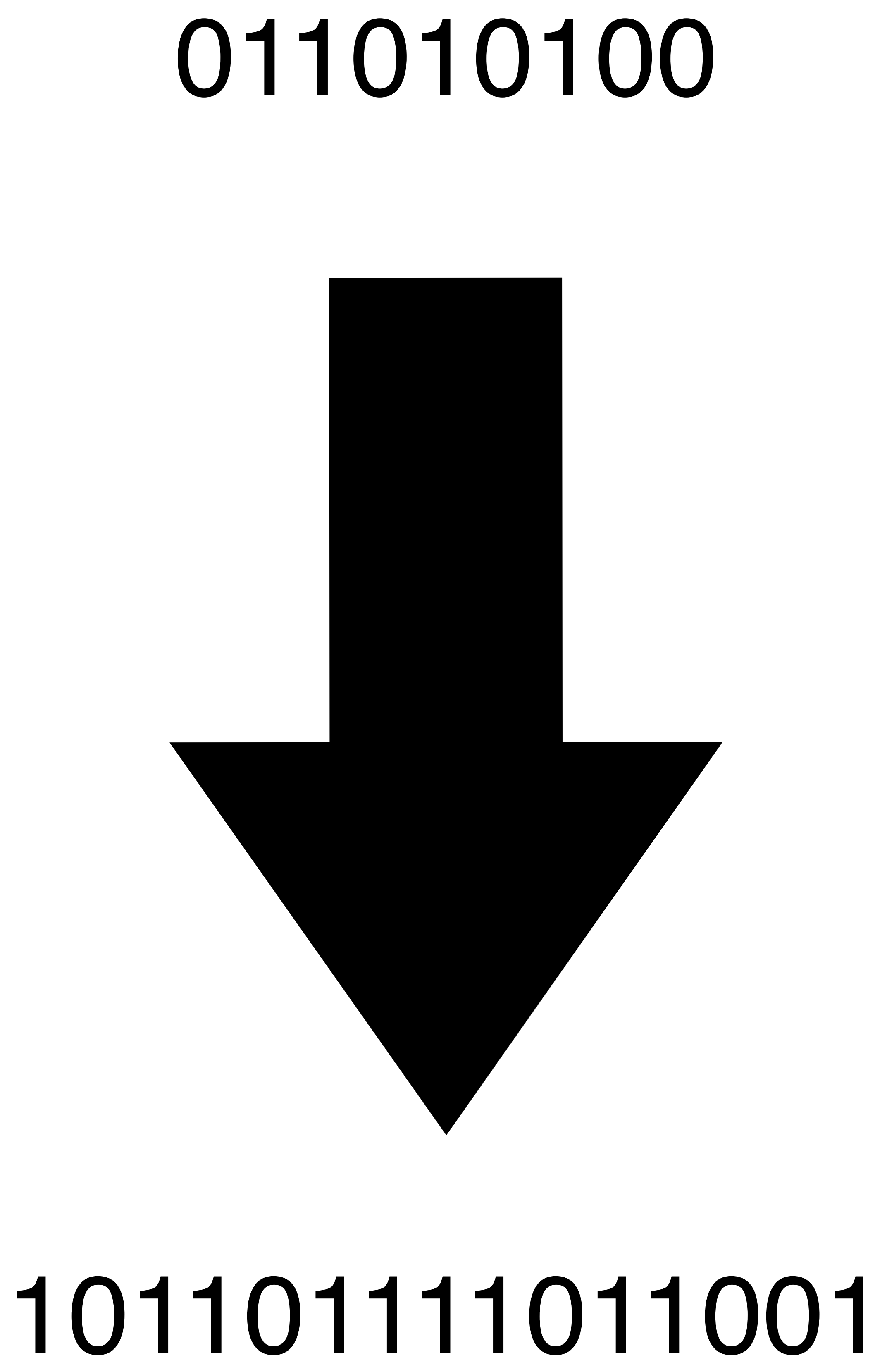
“If we want to encrypt more stuff, we need more randomness”



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Q: Can we turn a short random string into a long random string?

“If we want to encrypt more stuff, we need more randomness”

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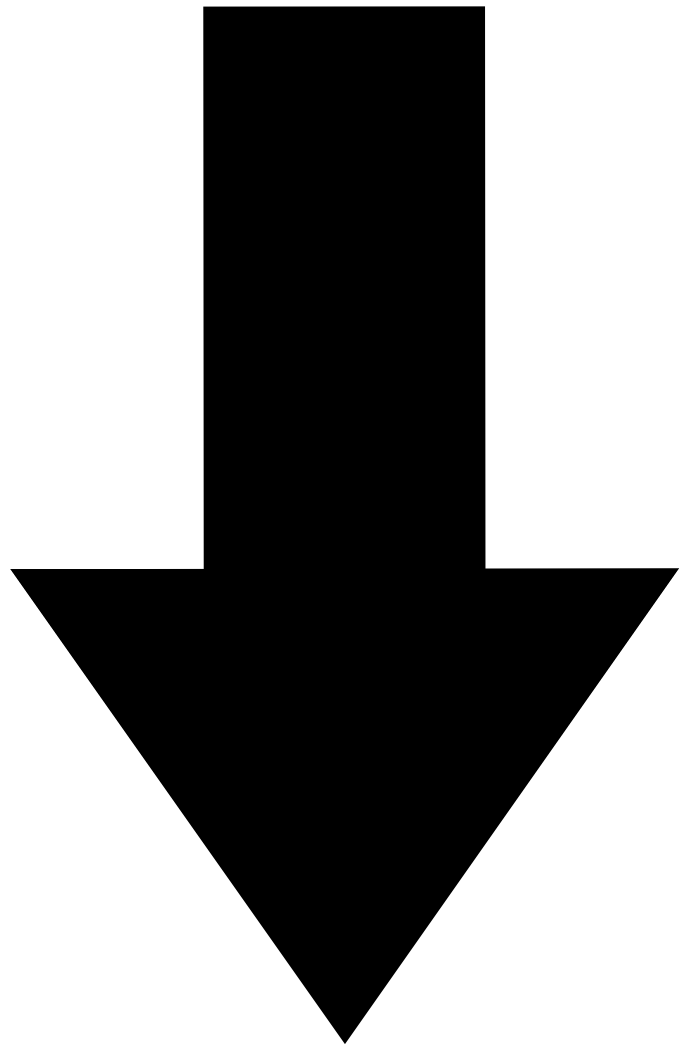

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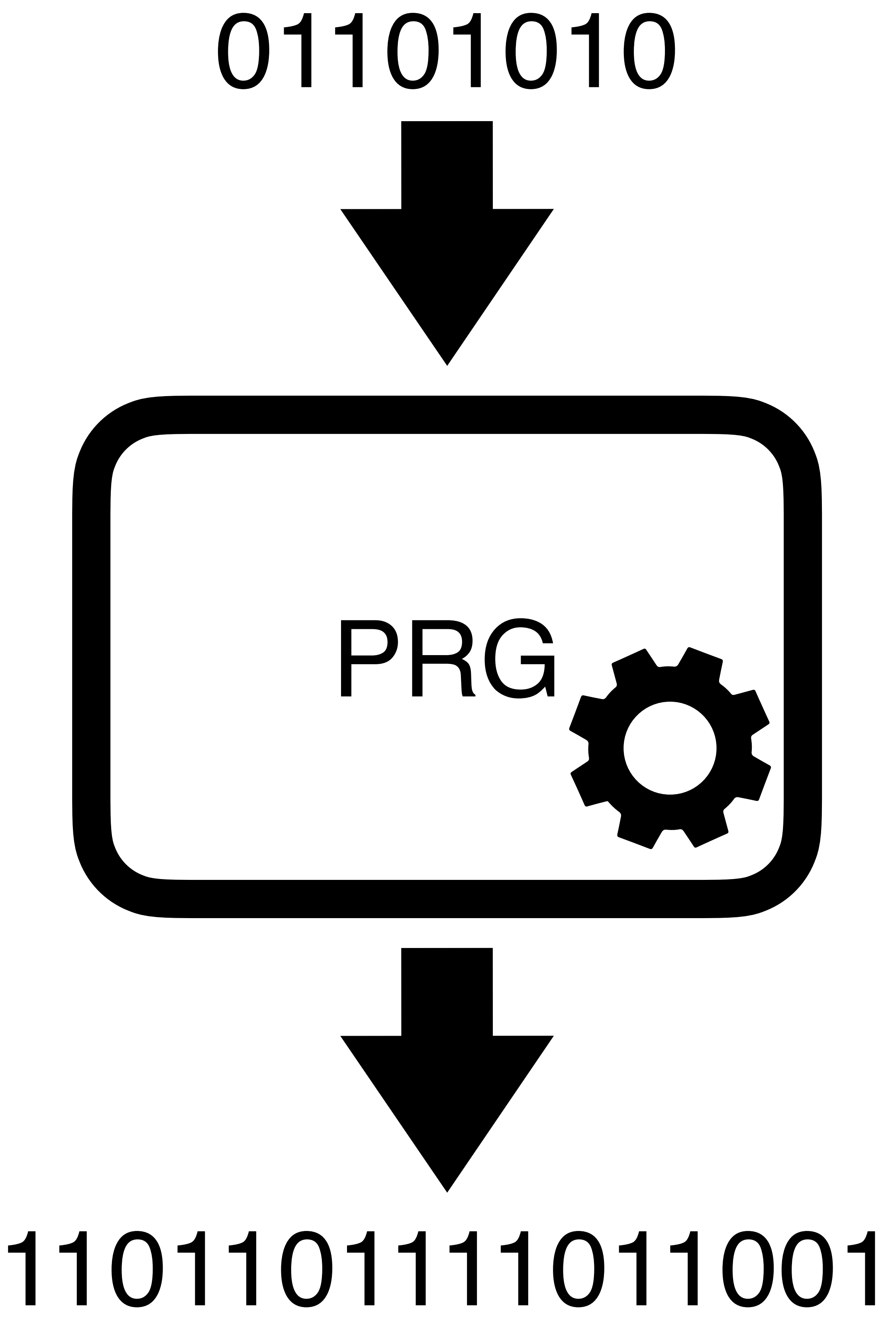
Q: Can we turn a short random string into a long random string?

A: No, this is provably impossible

"If we want to encrypt more stuff, we need more randomness"



Q: Can we turn a short random string into a long random string?

A: No, this is impossible

Q: Can we turn a short random string into a long string that <u>looks</u> random?

A: Yes! Use a pseudorandom generator!

Pseudorandom Generator (PRG)

A PRG is a function $G : \{0,1\}^n \rightarrow \{0,1\}^{n+s}$

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Informal: *"no program can tell the difference between the output of* G *and truly random strings"*

Hardness as a basis for cryptography

Security?

Informal: *"no program can tell the difference between the output of* G *and truly random strings"*

Modern Cryptography

State assumptions

Define security

Design system

Prove: if assumption holds, system meets definition

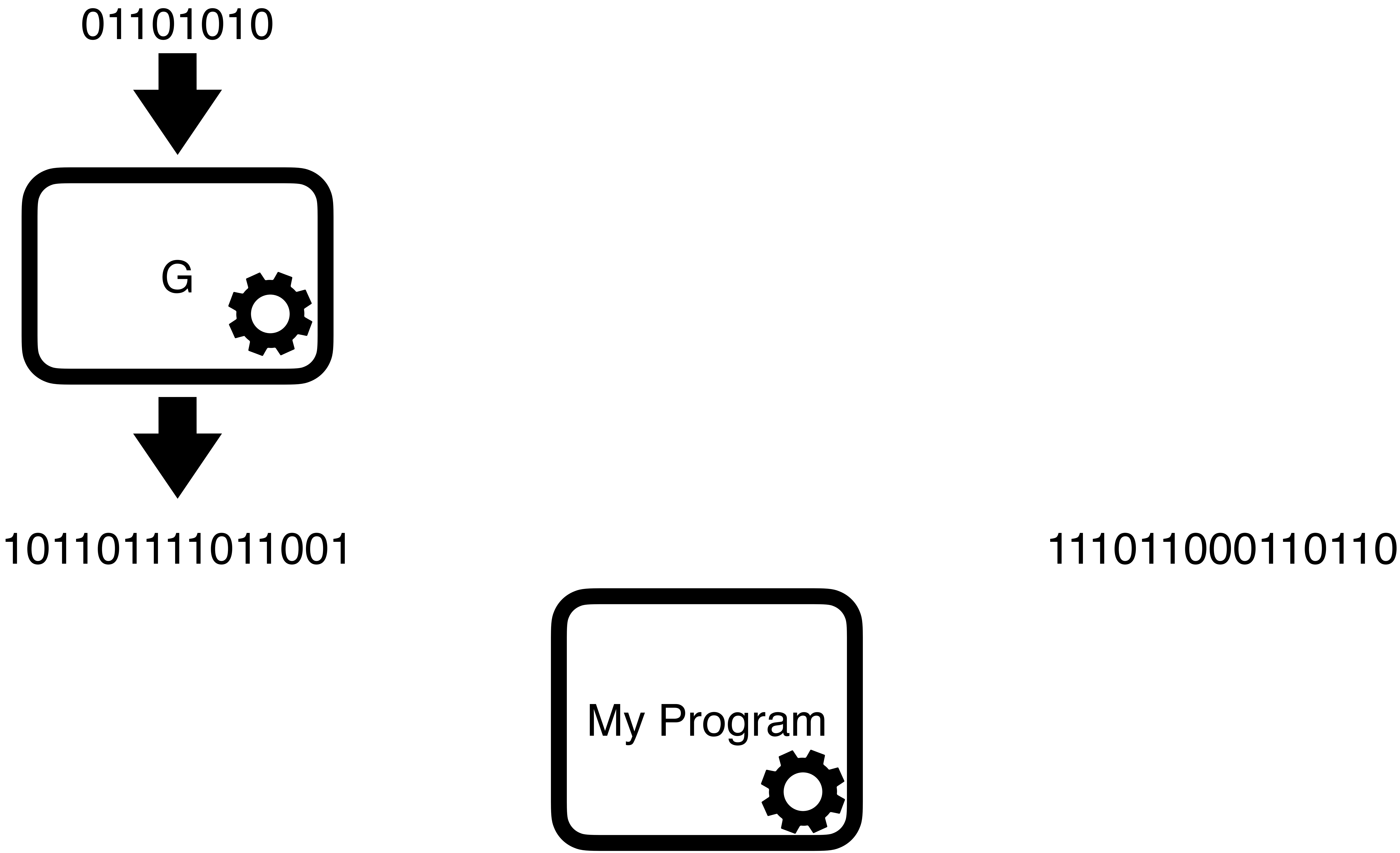
Modern Cryptography

State assumptions **PRGs exist**

Define security

Design system

Prove: if assumption holds, system meets definition



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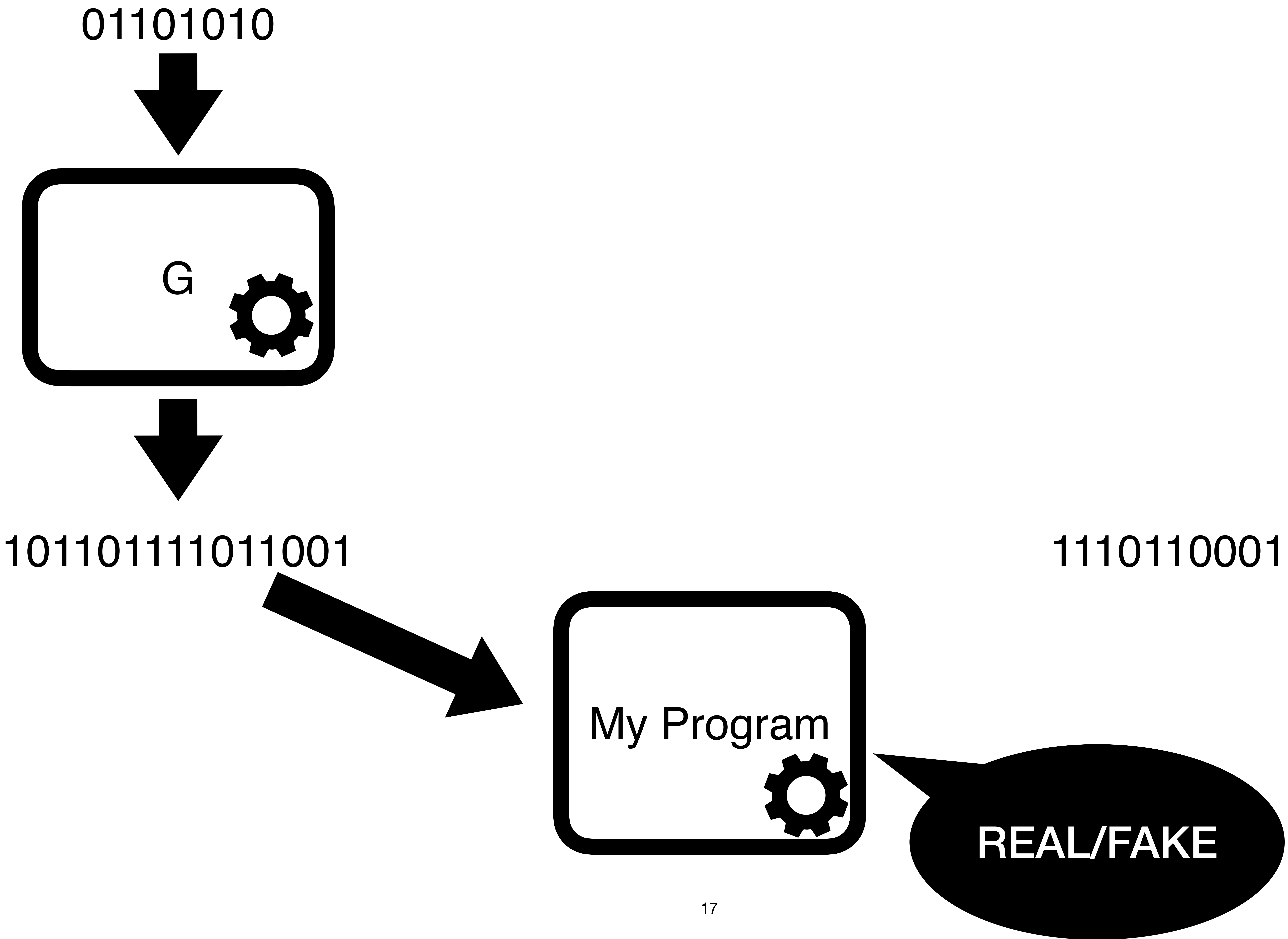
My Program



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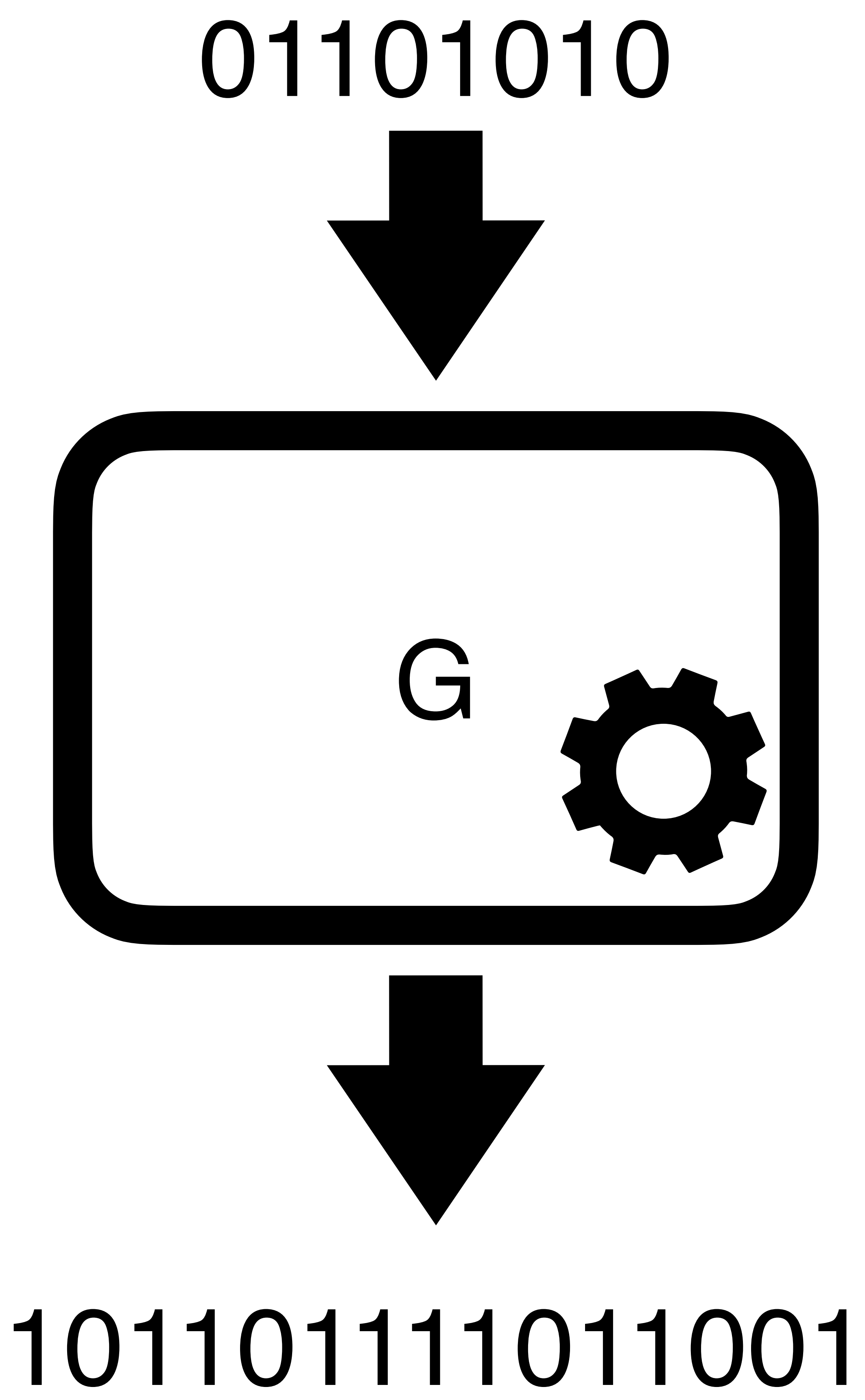
My Program

REAL/FAKE



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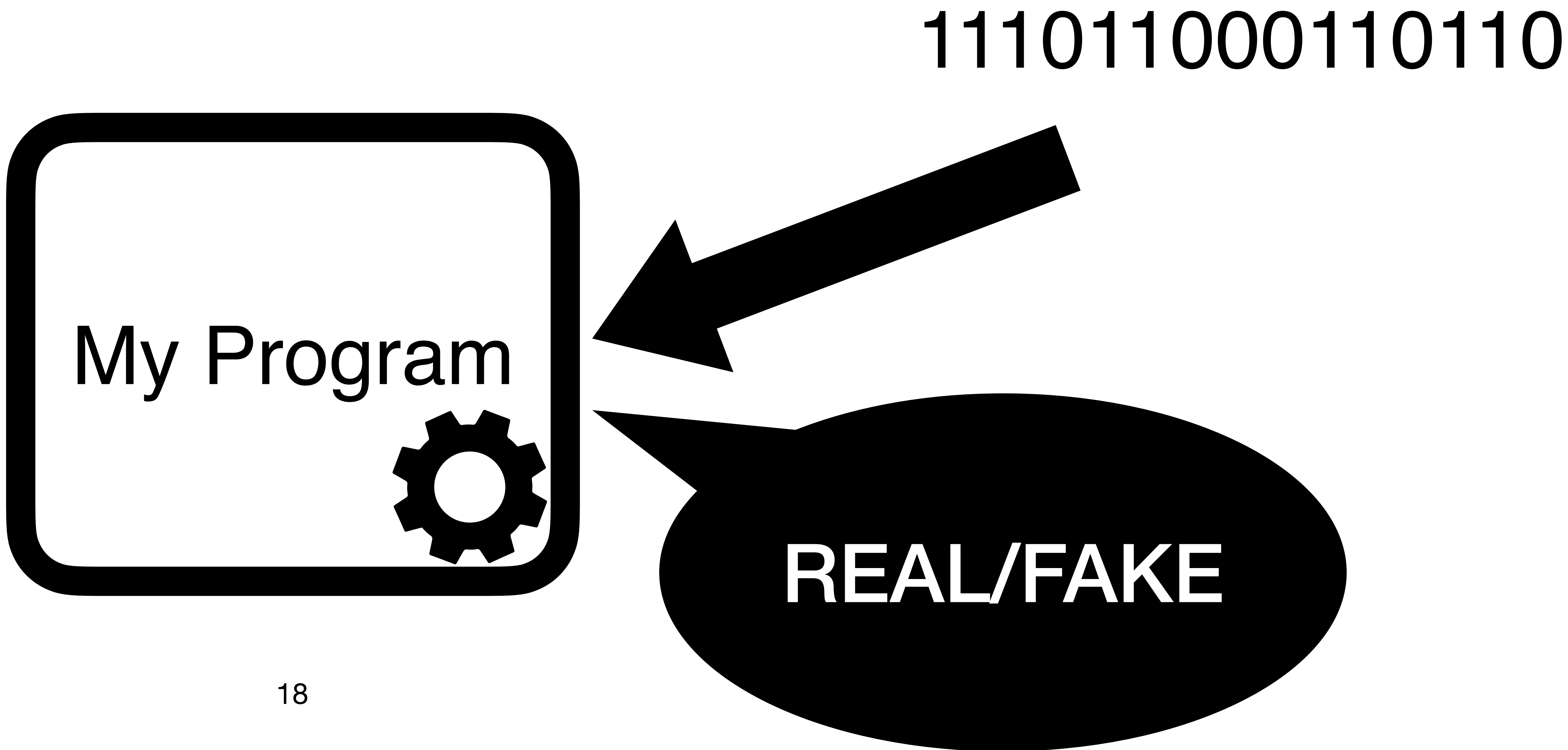

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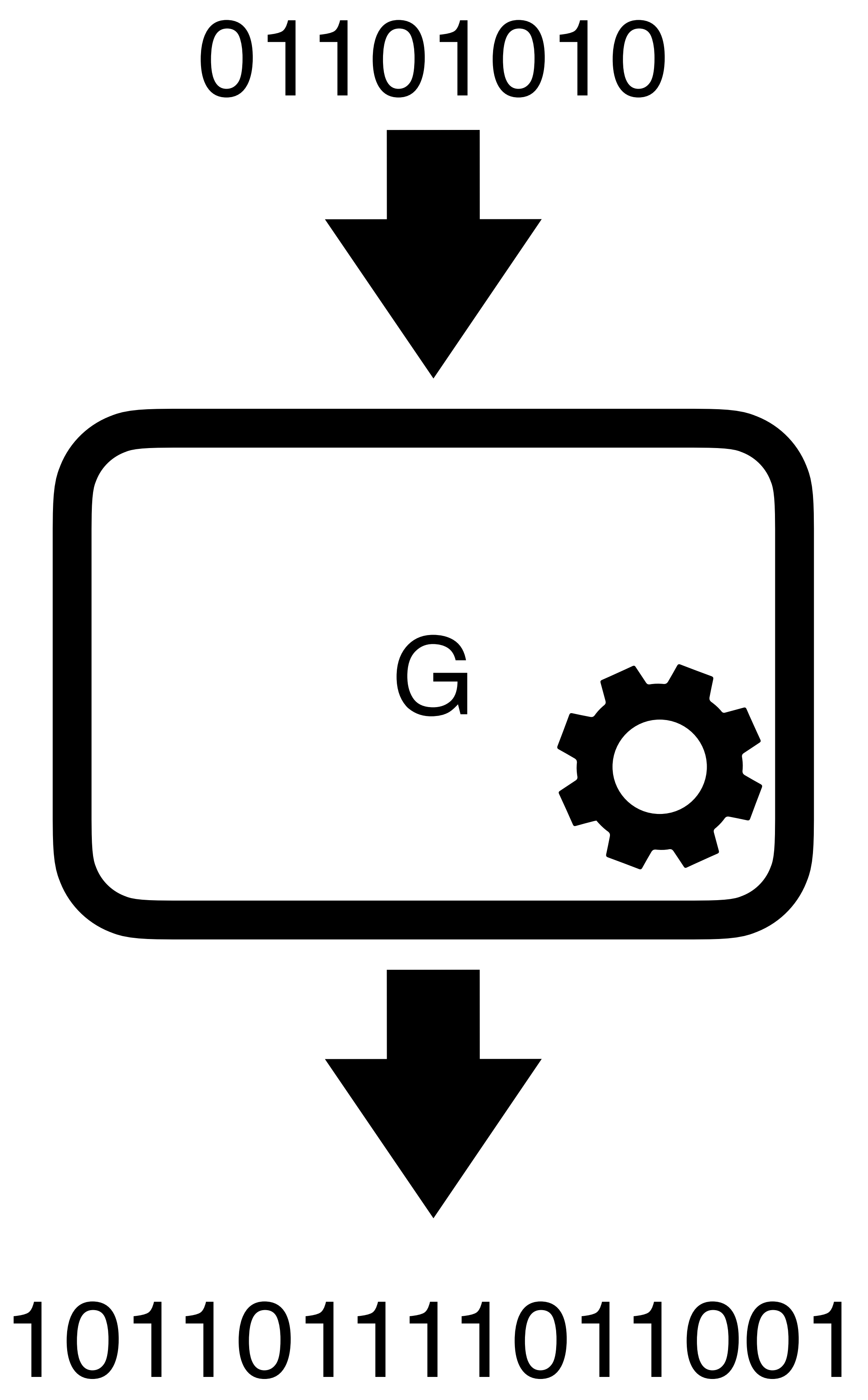
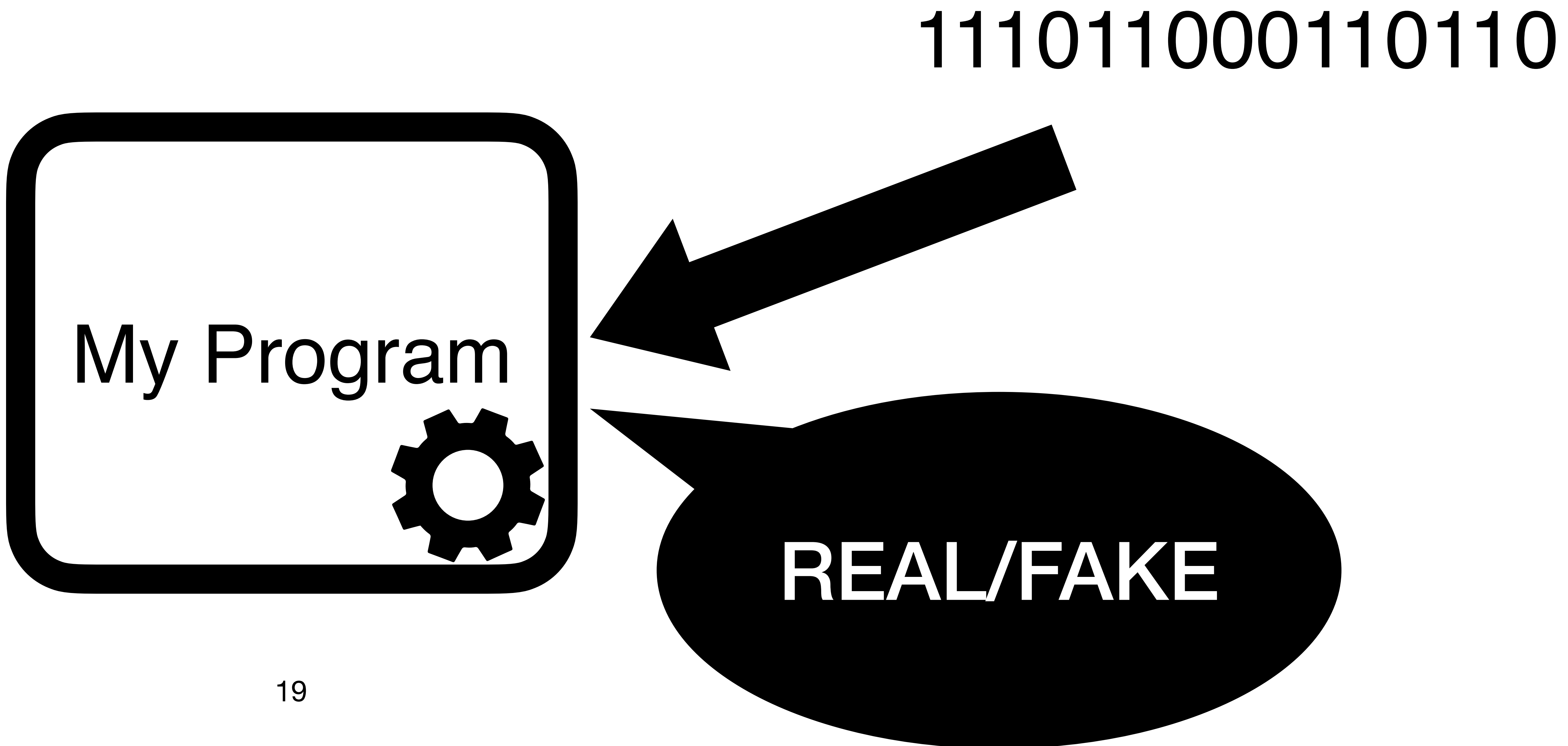


G is a PRG if *no* program can reliably win this game



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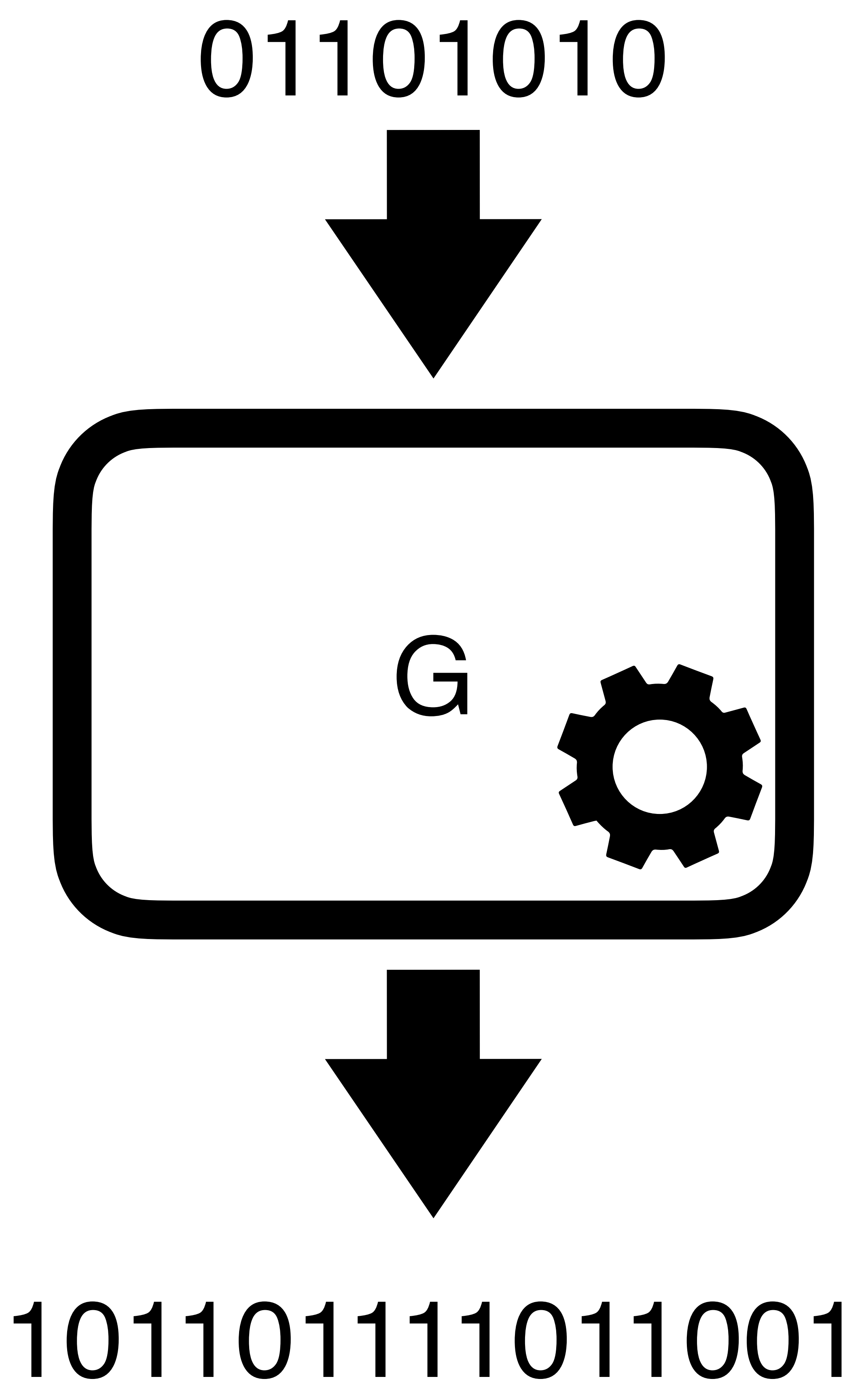

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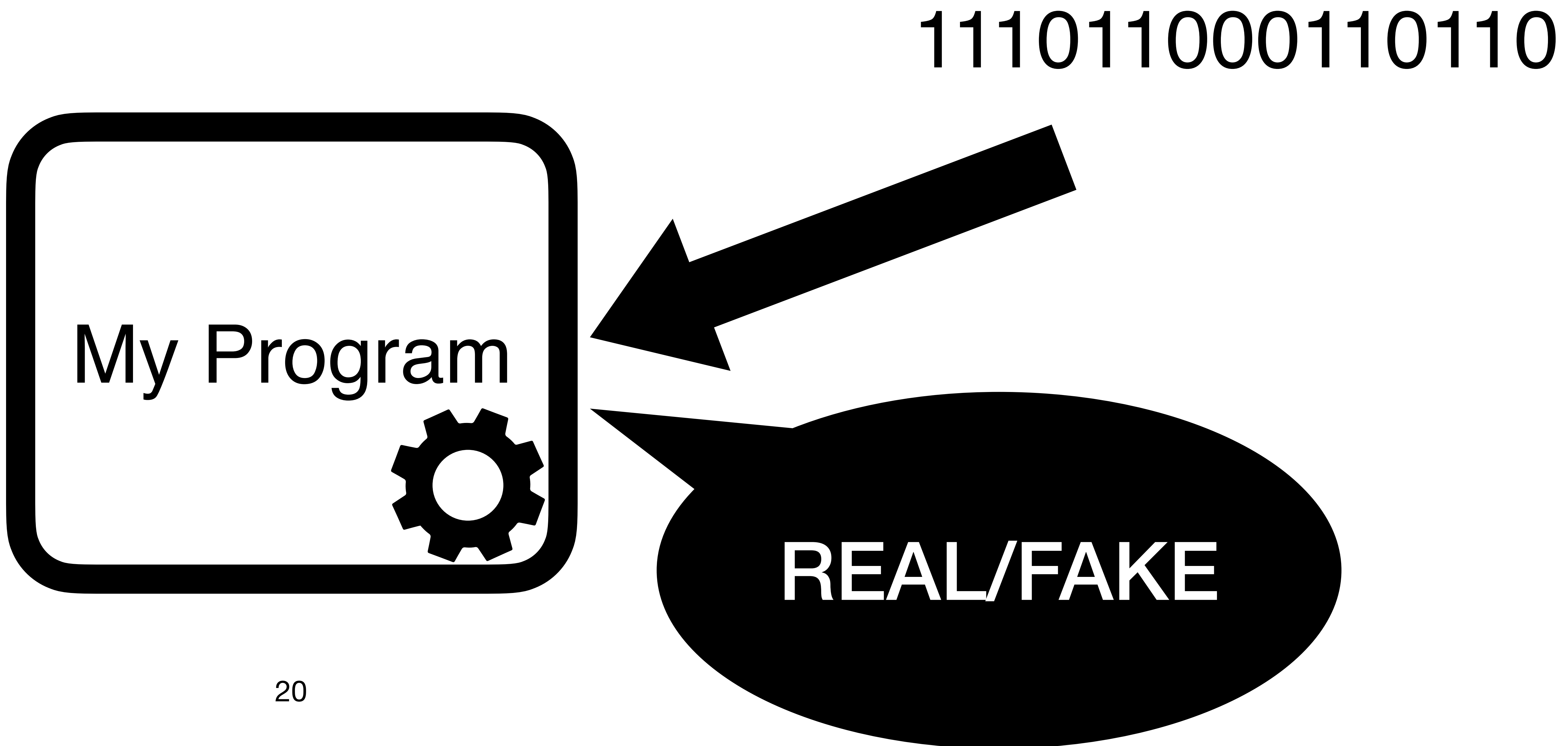
We believe that PRGs exist



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My Program

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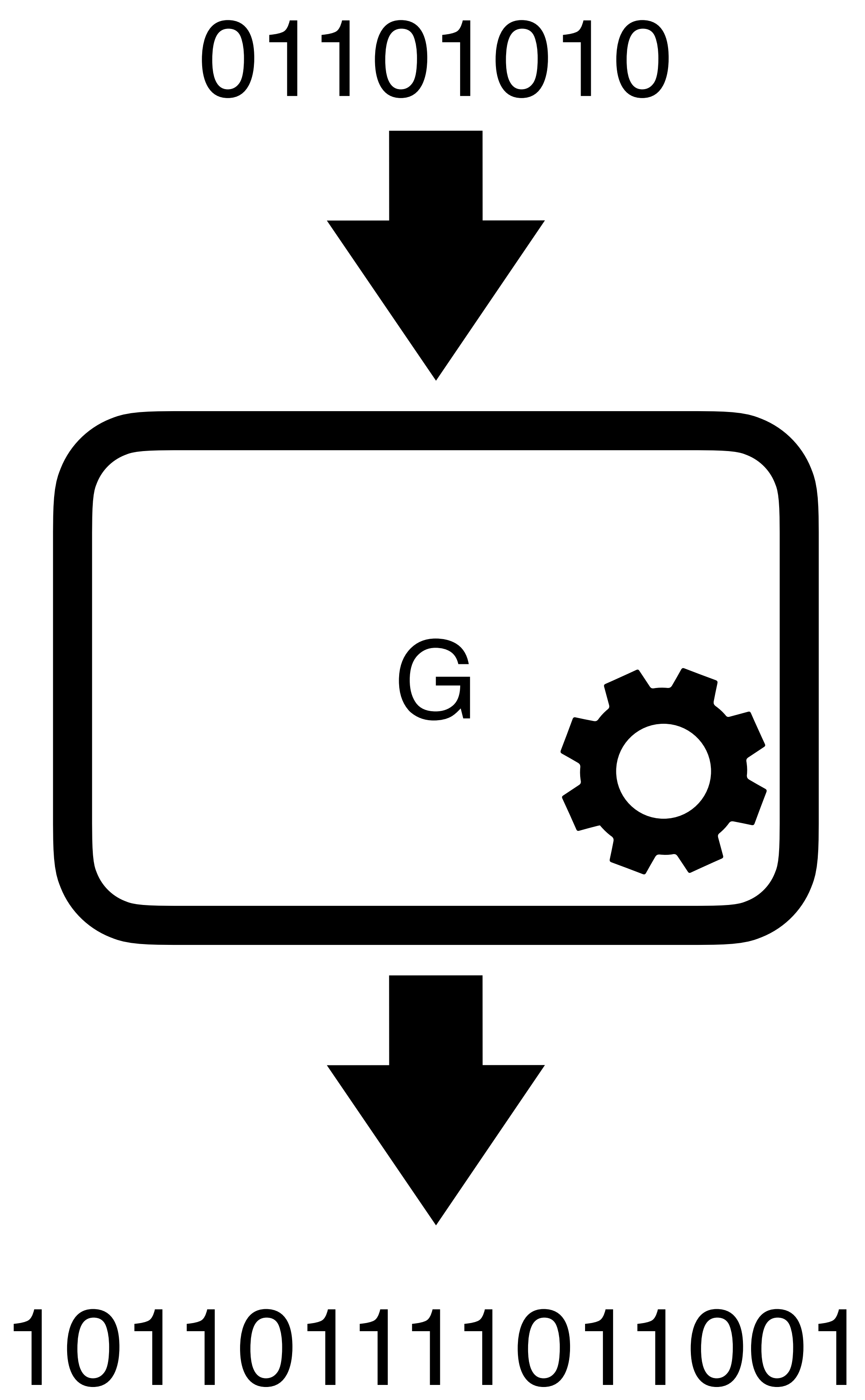

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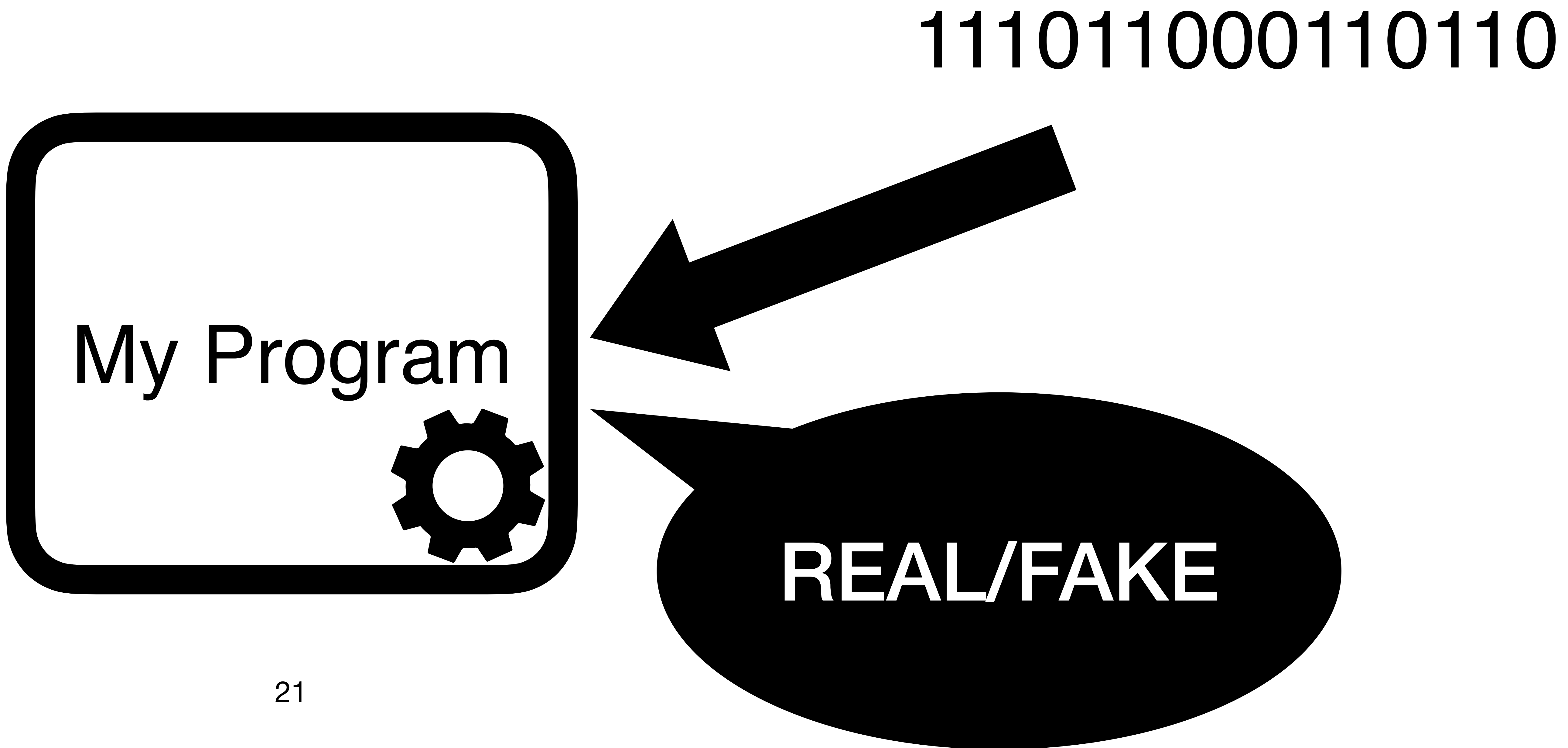

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If they do, $P \neq NP$

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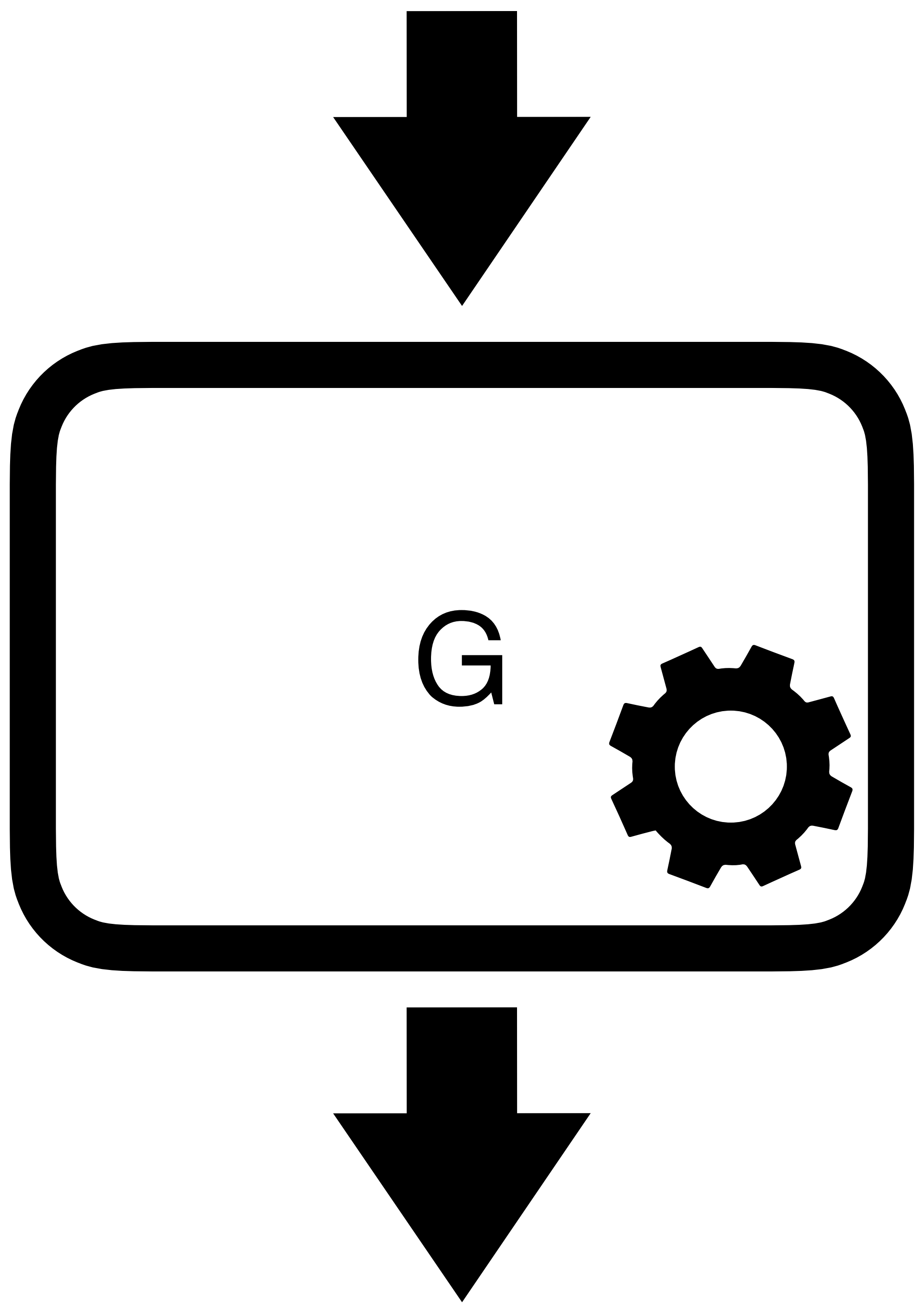

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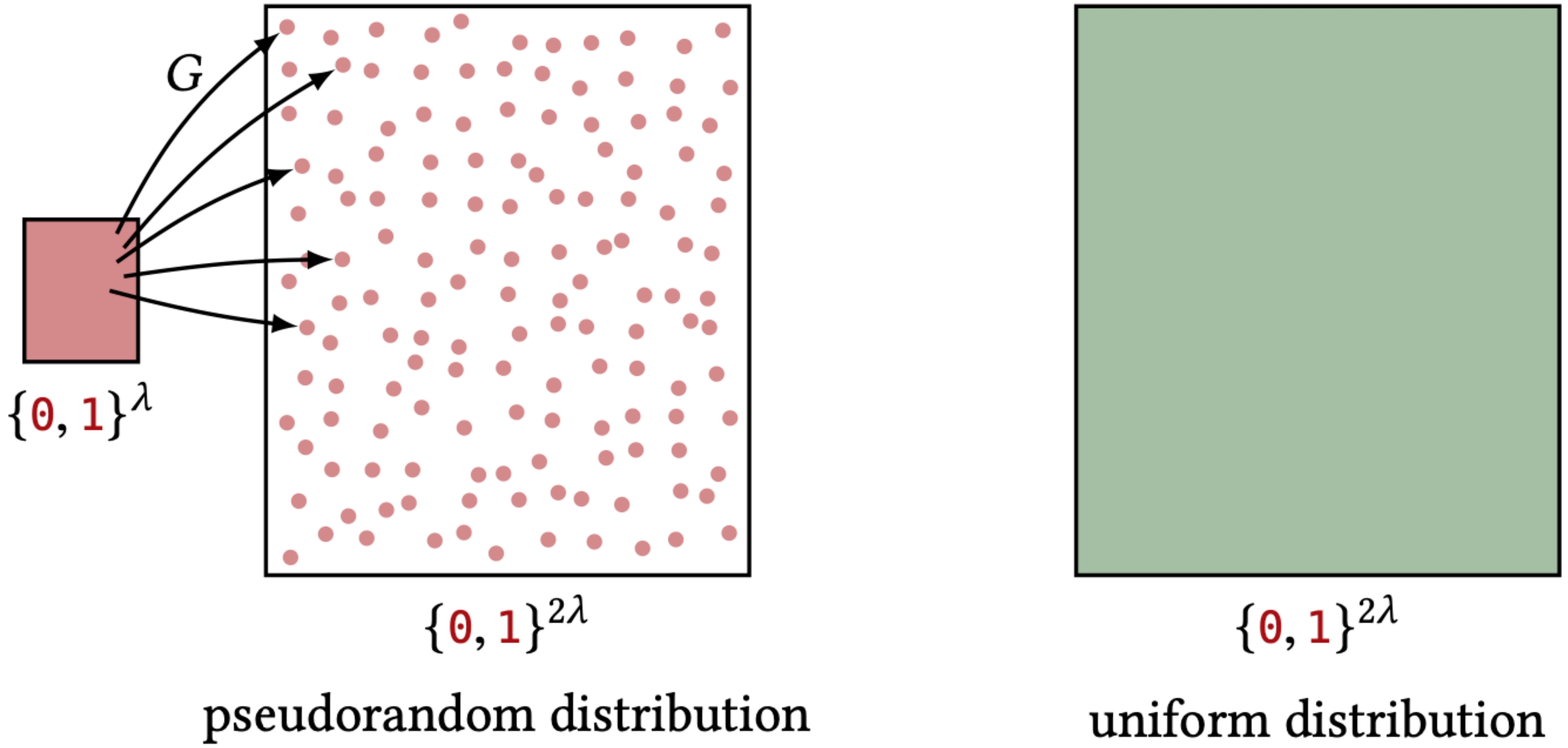

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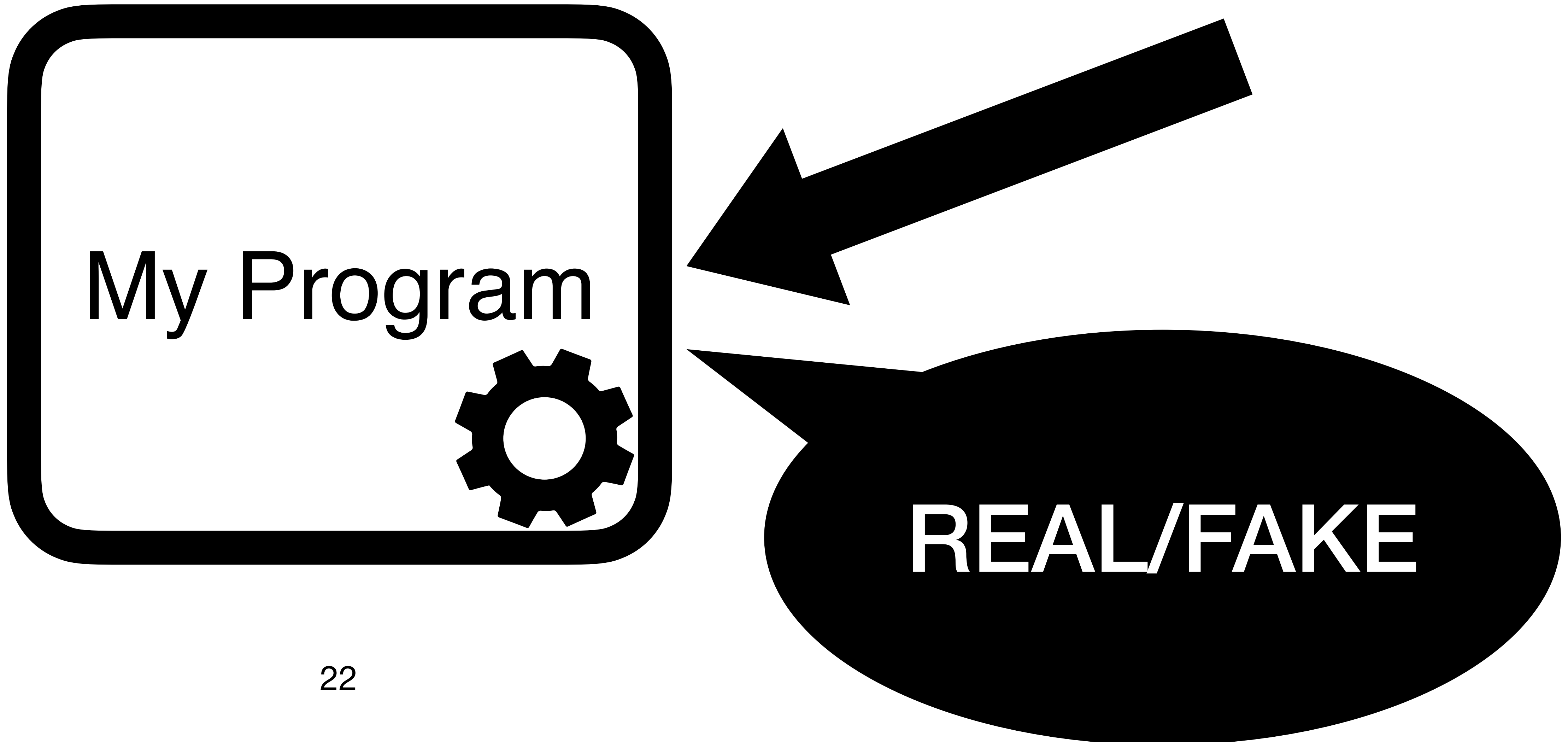

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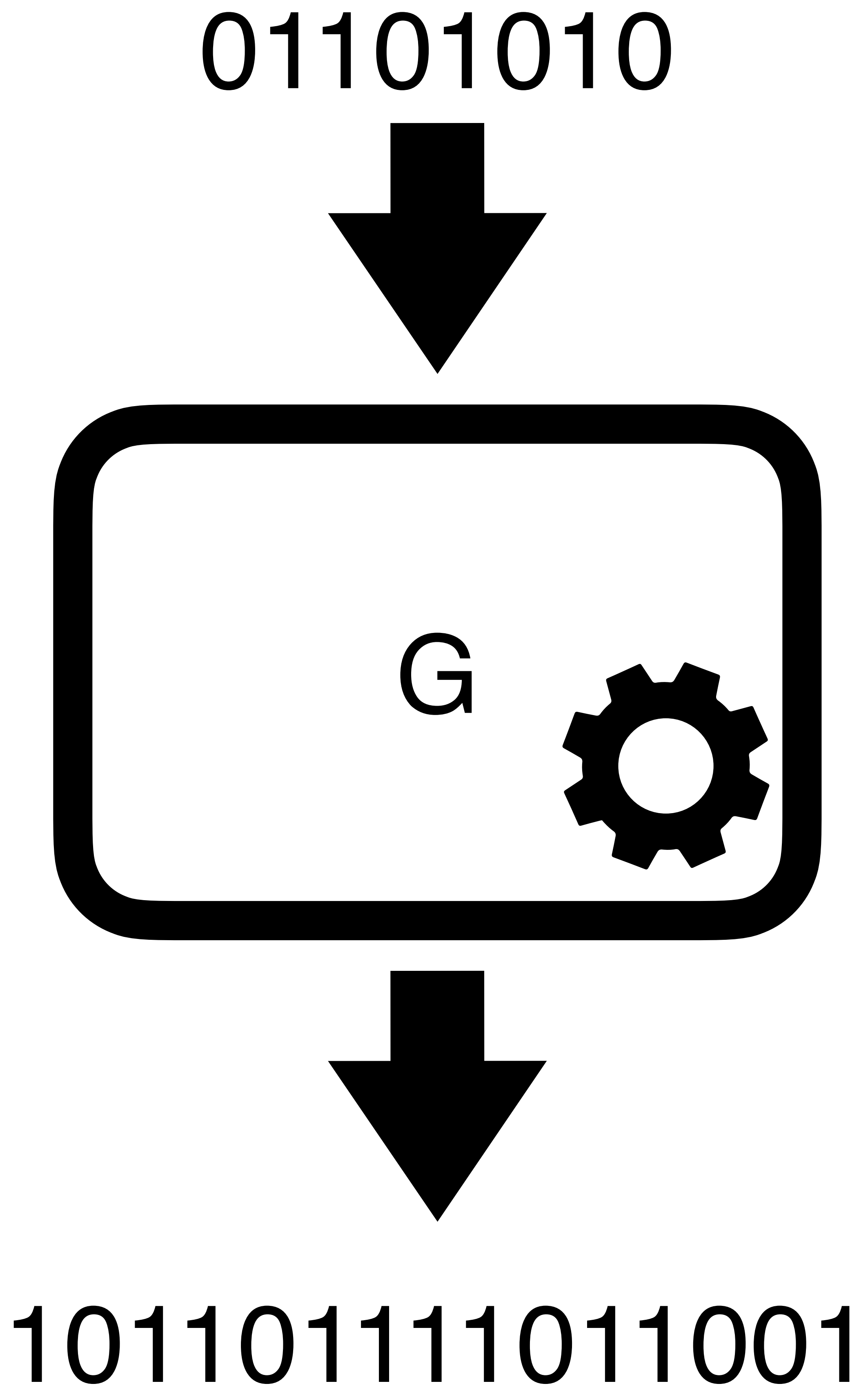
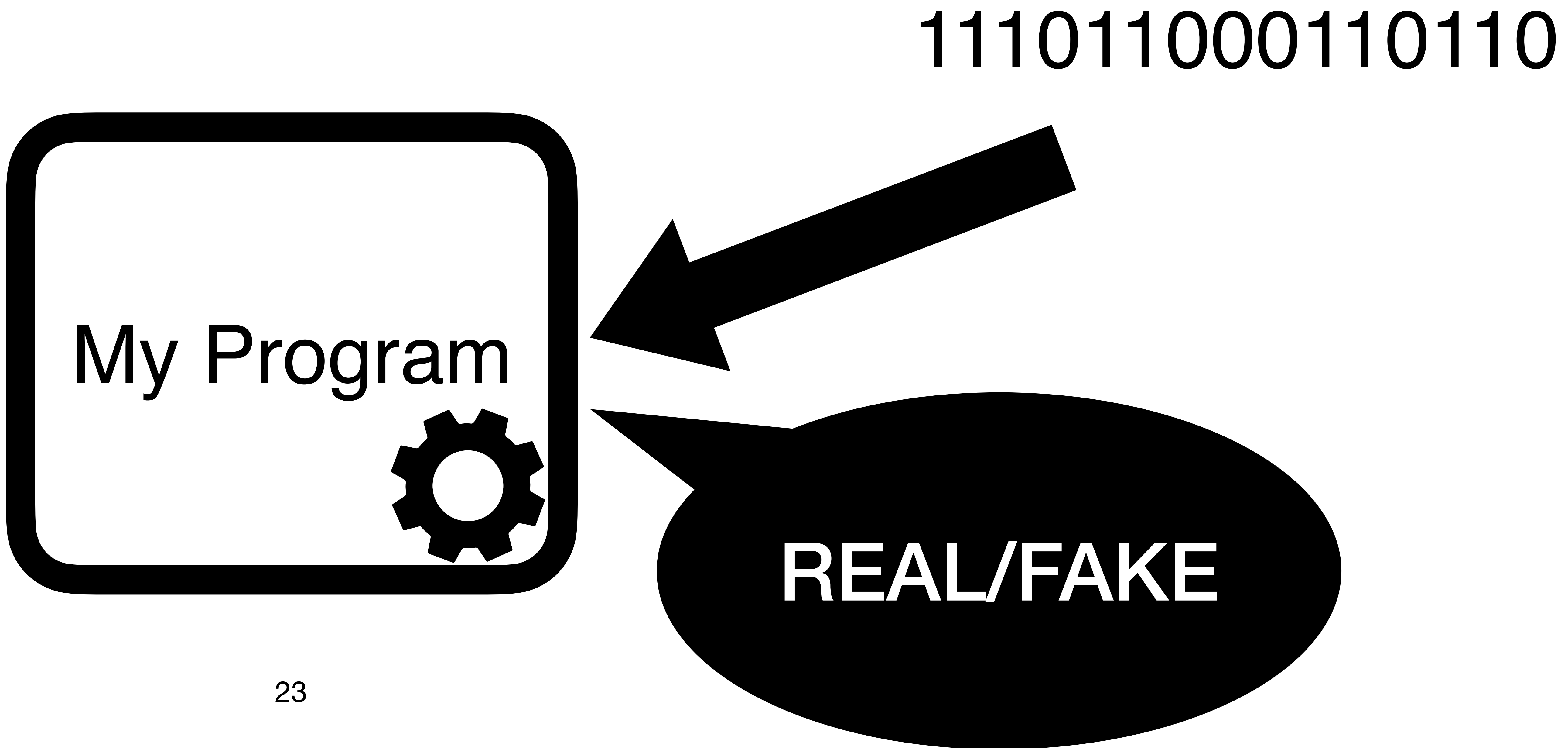
Goal: Make this more precise





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Negligible Function

A function μ is **negligible** *if for any positive polynomial p there exists an N such that for all $n > N$:*

$$\mu(n) < \frac{1}{p(n)}$$

“μ approaches zero really fast”

PRG security

Game 0

```
seed ←$ $\{0,1\}^n$
y := G(seed)
b := A(y)
```

Game 1

```
y ←$ $\{0,1\}^{n+s}$
b := A(y)
```

PRG security

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For *any* PPT algorithm A outputting a bit, the following quantity is **negligible** (in n):

$$|\Pr[\ b = 1 \mid \text{Game } 0\ ] - \Pr[\ b = 1 \mid \text{Game } 1\ ]|$$

PRG security

```
b ←$ $\{0,1\}$
if b = 0:
    seed ←$ $\{0,1\}^n$
    y := G(seed)
else
    y ←$ $\{0,1\}^{n+s}$
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For *any* PPT program A outputting a bit, the following quantity is **negligible** (in n):

$$\left| \Pr\left[\ b = b'\ \right] - \frac{1}{2} \right|$$

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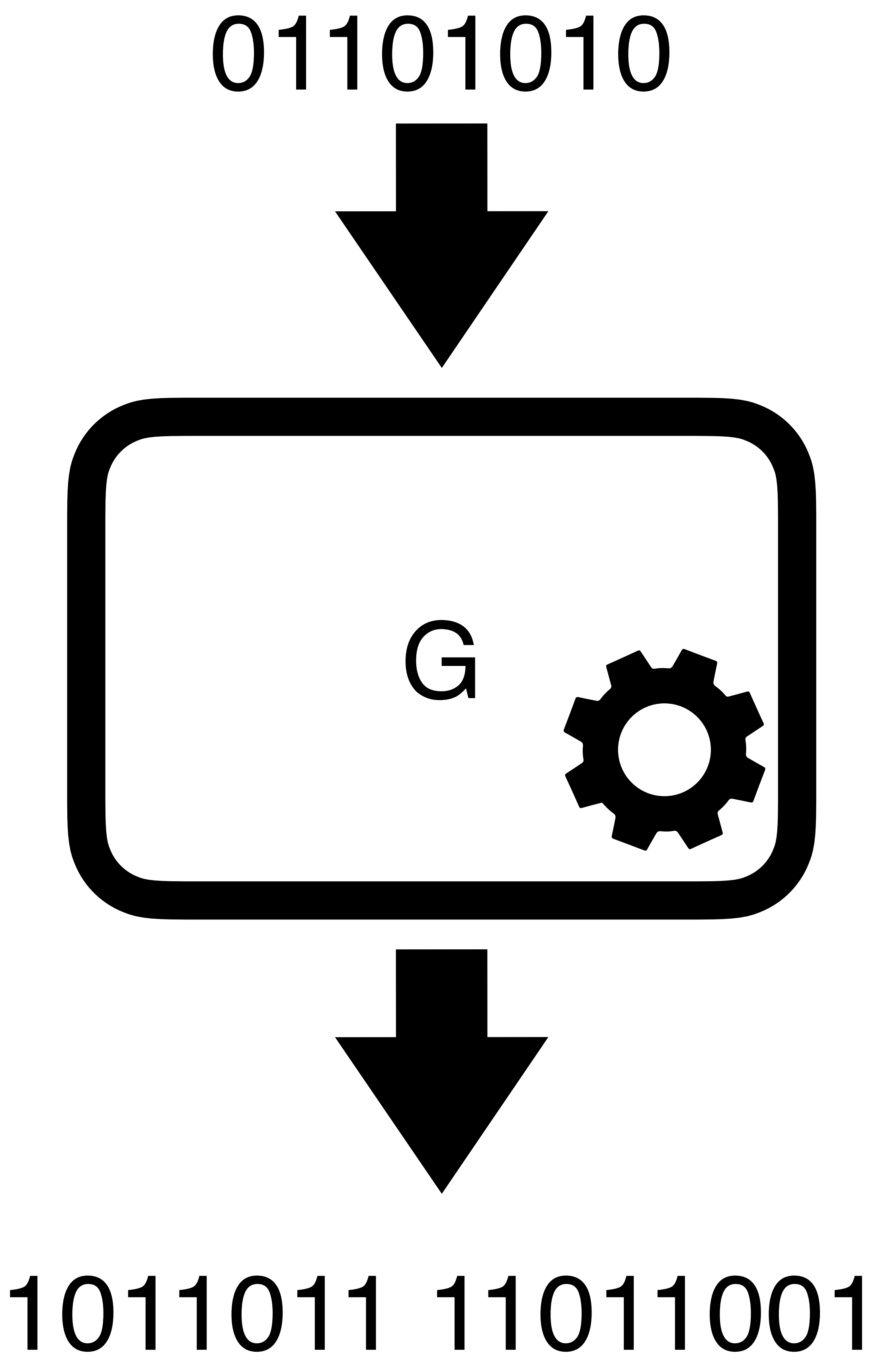
In other words, the best possible strategy is only negligibly better than simply guessing

Stretching the output of a PRG



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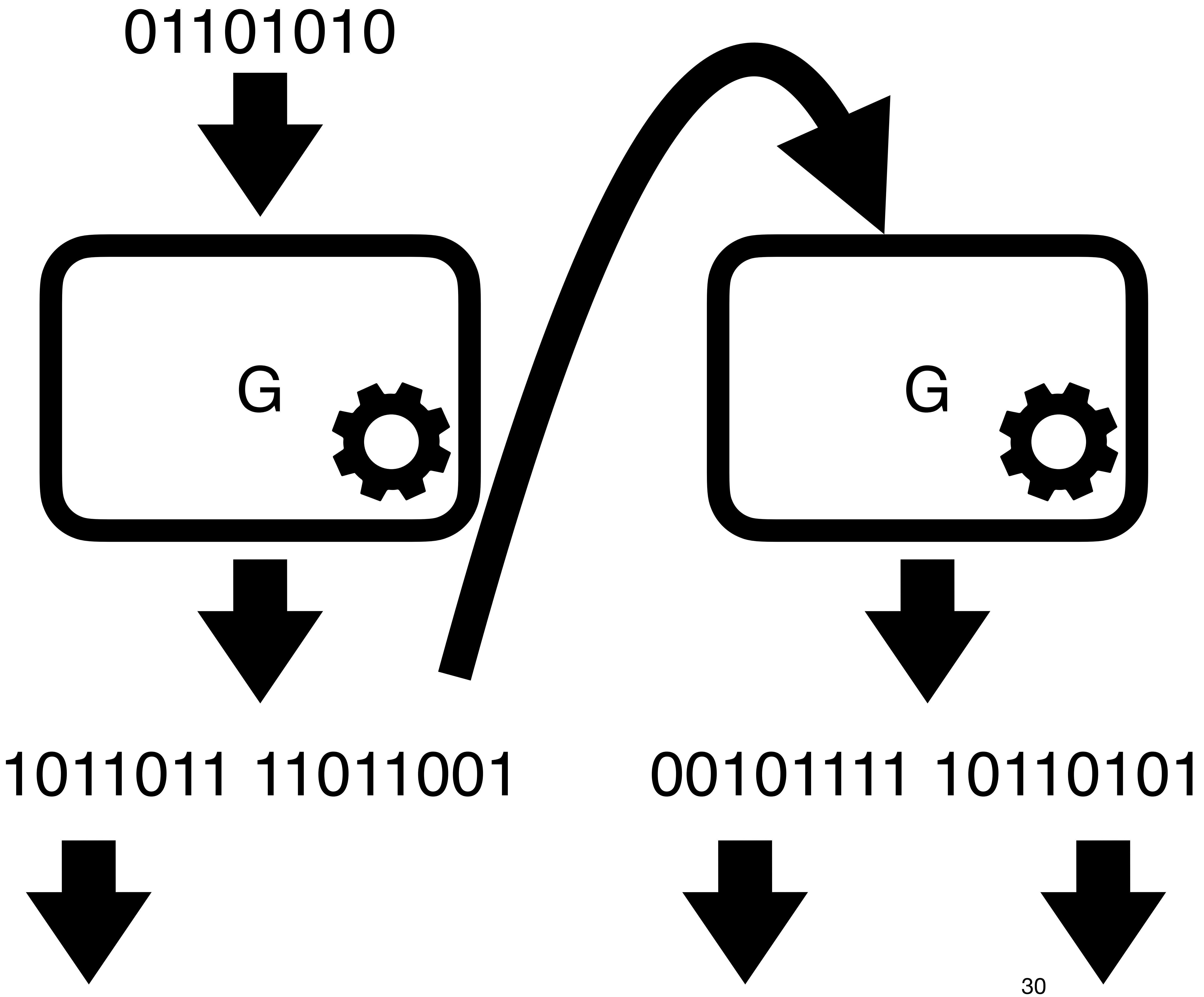
Stretching the output of a PRG



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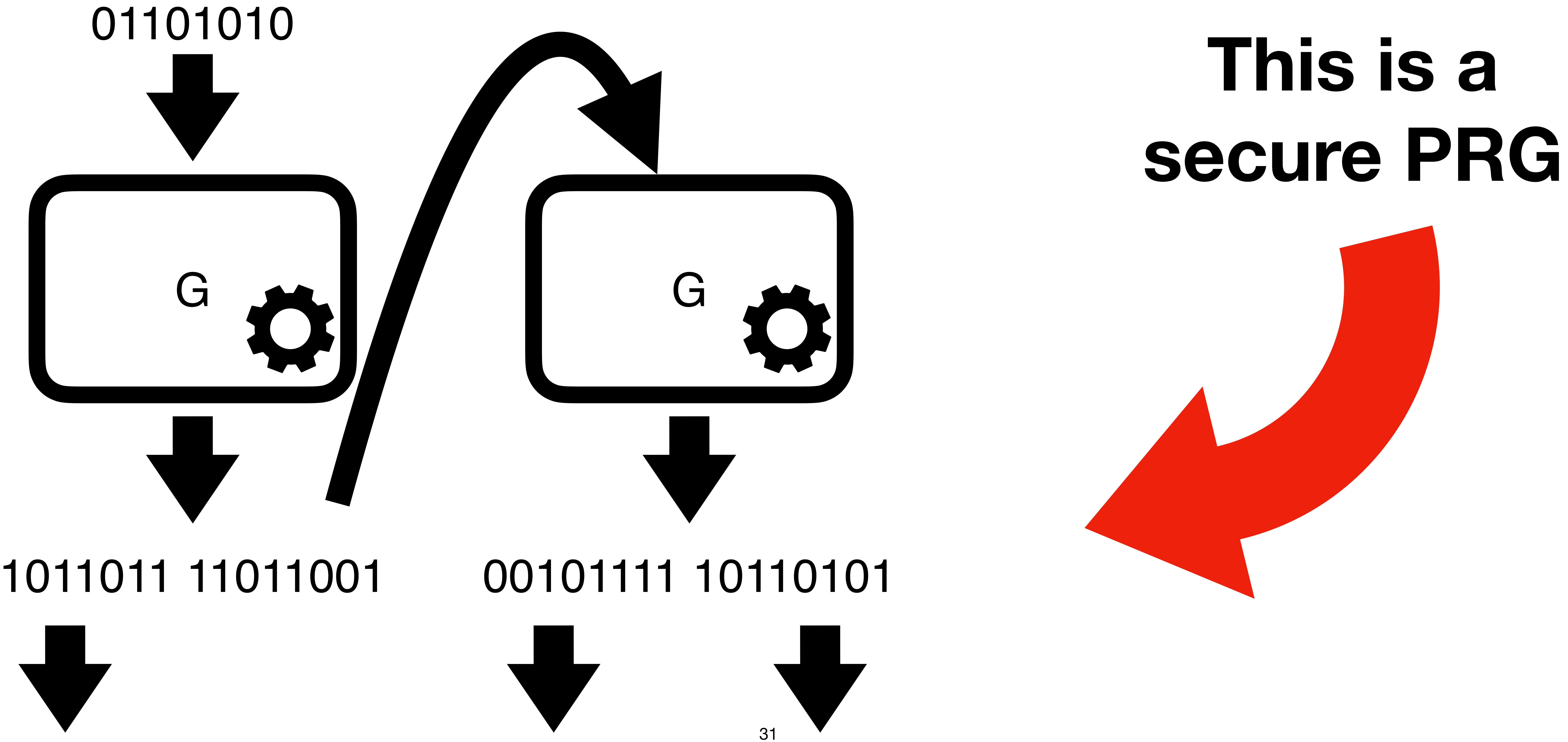

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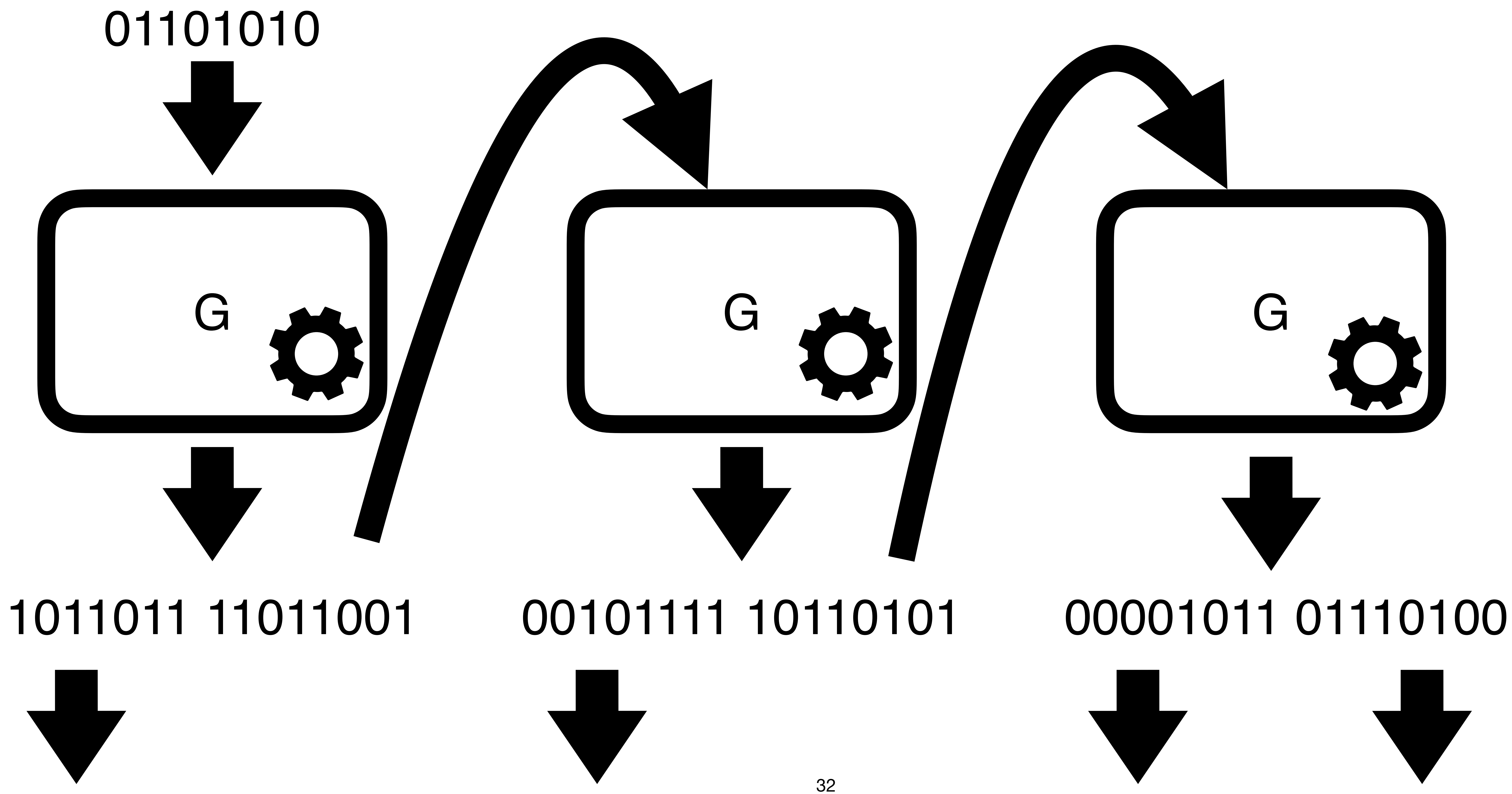
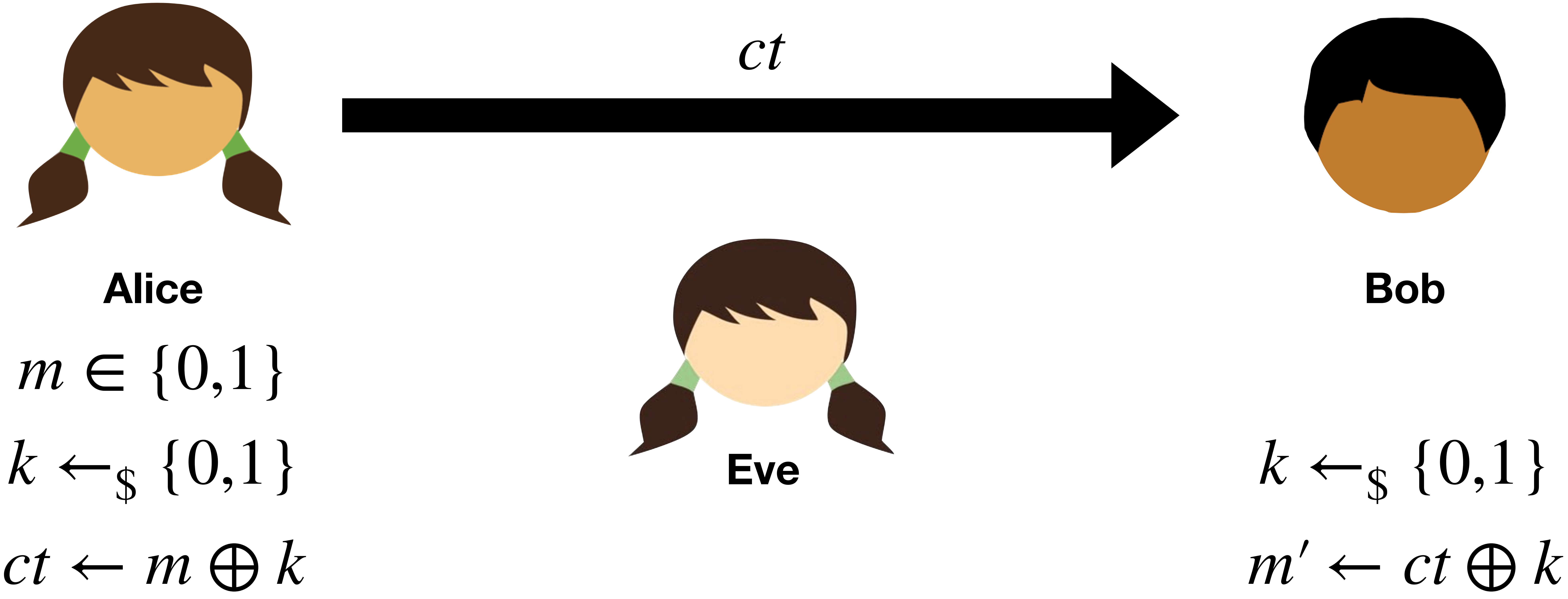
This is a secure PRG

Repeatable any polynomial number of times





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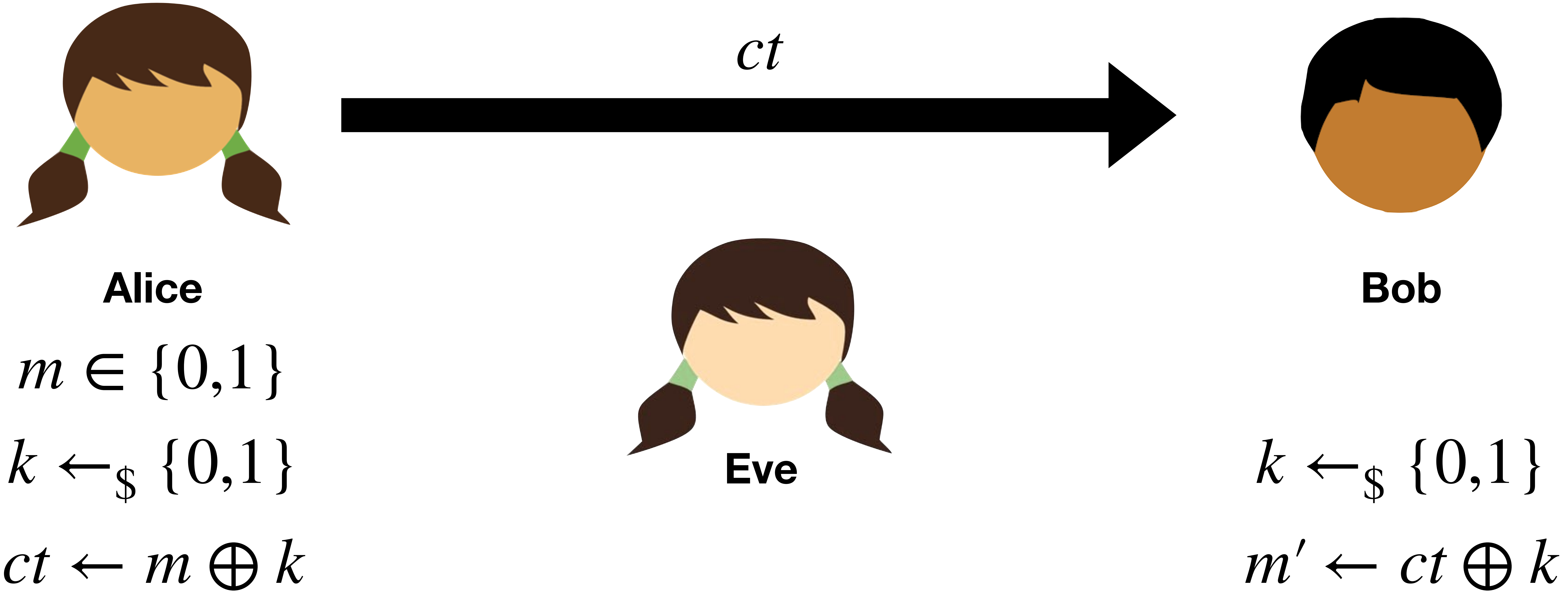

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Question: *what if Alice wants to send more than one bit?*

Answer: *Alice and Bob can exchange a short PRG seed, then expand it (effectively) indefinitely*

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Describe pseudorandomness/pseudorandom generators

Define negligible functions

Understand security of PRGs